Synchronizing the Noise-Perturbed Rössler Hyperchaotic System via Sliding Mode Control

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Z. Naturforsch. **66a,** 6 – 12 (2011); received September 4, 2009 / revised February 5, 2010

This paper investigates the synchronization problem between two unidirectionally-coupled Rössler hyperchaotic systems in the presence of noise perturbations. Sufficient conditions are obtained for synchronization by using a particularly simple linear sliding mode surface that is based on the sliding mode control concept. Only one controller function is needed to achieve synchronization in our present approach which makes it much easier to implement in contrast to many other synchronization schemes that require two or more controllers. Numerical simulation results are also included to illustrate the superior features of this new scheme.

Key words: Synchronization; Sliding Mode Control; Rössler Hyperchaotic System.

1. Introduction

In 1990, Pecora and Carroll [1] introduced the idea of synchronizing two identical chaotic systems that evolve from different initial conditions. It soon transpires after their pioneering work that chaos synchronization has great potentials for applications in diverse areas of science and engineering including communications security, information processing, and many applications to chemical and biological systems (for example, in regulating human heartbeats, and so on). Many different methods have been proposed for synchronizing chaotic systems since the days of Pecora and Carroll such as the linear and nonlinear feedback method [2], the time-delay feedback method [3], the adaptive method [4], the impulsive method [5], and the function projective synchronization method that can be applied to both continuous and discrete-time chaotic systems [6, 7]. Fractional-order chaotic systems can also be synchronized using the Laplace transform theory [8 – 10]. However, most of the existing synchronization methods are only valid for chaotic systems that are free from noise perturbations and contain only low dimensional attractors that are characterized by one positive Lyapunov exponent. Clearly, noise, in the form of matched and mismatched disturbances, exist in real-world chaotic systems. Furthermore, it is obvious that communication security would be greatly enhanced by the use of higher-dimensional chaotic systems that possess multiple positive Lyapunov exponents because of the complex dynamics that is generated. Indeed, the physical significance of hyperchaotic systems cannot be over-emphasized because they can be used as models for many scientific problems and phenomena [11]. Likewise, the synchronization of hyperchaotic systems with noise perturbations is of utmost importance.

In recent years, the sliding mode control method has become a popular nonlinear control strategy, even though it is known that this method has its drawbacks. For instance, the traditional sliding mode control law is unable to predict the time that it takes to reach the sliding surface that assures system robustness in the approaching phase. Nevertheless, sliding mode control has recently been applied extensively to chaos control and synchronization [12 – 16], with one particularly successful instance of synchronization in [14]. Indeed, in this paper, we introduce a simple linear sliding mode control method that ensures the synchronization of two unidirectionally-coupled Rössler hyperchaotic systems in the presence of noise perturbations. Furthermore, synchronization can be realized by only one controller function which makes the scheme much easier to implement. As linear surfaces are rarely used for Rössler systems, our proposed method represents a contribution to developments in this direction.

2. The Rössler Hyperchaotic Systems

The Rössler system, first introduced by Rössler [17], is a simple and prototypical equation with chaotic behaviour for the Lorenz model of turbulence that contains just one (second-order) nonlinearity in one variable. Subsequently, Rössler [18] also proposed a four-variable oscillator (which later came to be called the Rössler hyperchaotic system by other researchers) that contains only a single nonlinear term of quadratic type but which produces chaos with two distinct directions of hyperbolic instability on the attractor. The Rössler hyperchaotic system is given by the set of nonlinear differential equations

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + a x_2 + x_4, \\ \dot{x}_3 &= b + x_1 x_3, \\ \dot{x}_4 &= -c x_3 + d x_4, \end{aligned} \tag{1}$$

which has a hyperchaotic attractor for the positive system parameters $a = 0.25$, $b = 3$, $c = 0.5$, and $d = 0.05$. The controlled Rössler hyperchaotic system with noise perturbation is given by

$$\begin{aligned} \dot{y}_1 &= -y_2 - y_3 + d_1, \\ \dot{y}_2 &= y_1 + a y_2 + y_4 + d_2, \\ \dot{y}_3 &= b + y_1 y_3 + d_3 + v, \\ \dot{y}_4 &= -c y_3 + d y_4 + d_4, \end{aligned} \tag{2}$$

where d_1, d_2, and d_4 are mismatched disturbances and d_3 is a matched disturbance (see [19], P578). Assuming that these disturbances are all bounded in C^1 (i.e., that $\|d_i\|_1 \le \delta < 1$ $(i = 1,2,3,4)$ for some constant δ, where $\|\cdot\|_1$ is a usual norm in C^1 given by $\|u(t)\|_1 = \sup_{t\in\mathbb{R}}\{|u(t)| + |u'(t)|\}$), the aim of this paper is to design a controller v such that the controlled Rössler hyperchaotic system with disturbance (2) is synchronous with the master system (1).

3. Synchronizing the Rössler Hyperchaotic Systems with Noise Perturbation

In this section, we shall design the sliding mode controller that is needed to synchronize the Rössler hyperchaotic systems with disturbance. First, we define the error signal as $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$, and $e_4 = y_4 - x_4$. Then, by subtracting (1) from (2), we have the error state dynamics equations

$$\begin{aligned} \dot{e}_1 &= -e_2 - e_3 + d_1, \\ \dot{e}_2 &= e_1 + a e_2 + e_4 + d_2, \\ \dot{e}_3 &= x_1 e_3 + x_3 e_1 + e_1 e_3 + d_3 + v, \\ \dot{e}_4 &= -c e_3 + d e_4 + d_4, \end{aligned} \tag{3}$$

Now, we choose a suitable sliding mode surface $S = 0$ of the form

$$S = e_3 + p(e_2 + e_4) \tag{4}$$

for some real constant p whose value has yet to be assigned. We have the following result.

Theorem 1. If the controller v is selected as

$$\begin{aligned} v = {} & -x_1 e_3 - x_3 e_1 - e_1 e_3 - p e_1 - a p e_2 \\ & + c p e_3 - p(1+d) e_4 - k\,\mathrm{sgn}(S), \end{aligned} \tag{5}$$

where p is a constant and $k > (2|p|+1)\delta + 1$, then the states of the error system (3) will approach the sliding mode surface $S = 0$ in finite time.

Proof: Choose $V = S^2$ as the Lyapunov function of the system (3). Then the first derivative of V along the solutions of system (3) is

$$\begin{aligned} \dot{V} &= 2S(\dot{e}_3 + p\dot{e}_2 + p\dot{e}_4) \\ &= 2S[x_1 e_3 + x_3 e_1 + e_1 e_3 + p e_1 + a p e_2 \\ &\quad - c p e_3 + p(1+d) e_4 + d_3 + p(d_2 + d_4) + v]. \end{aligned}$$

In real applications, the disturbances d_1, d_2, d_3, and d_4 are unknown. So by (5), when $k > (2|p|+1)\delta + 1$, we have

$$\begin{aligned} \dot{V} &= 2S[d_3 + p(d_2 + d_4) - k\,\mathrm{sgn}(S)] \\ &\le 2|S|[(2|p|+1)\delta - k] \le -2|S| = -2V^{\frac{1}{2}} \end{aligned}$$

which implies that $V^{\frac{1}{2}}(t) \le V^{\frac{1}{2}}(0) - t$, $t \in [0, t_s]$ and $V(t) = 0$ when $t \ge t_s = V^{\frac{1}{2}}(0)$. The proof is thus complete.

Next we give the synchronization analysis, noting that it suffices to analyze the error system on the sliding mode surface (because of Theorem 1). On the sliding mode surface $S = 0$, the error system reads:

$$\begin{aligned} \dot{e}_1 &= (p-1) e_2 + p e_4 + d_1, \\ \dot{e}_2 &= e_1 + a e_2 + e_4 + d_2, \\ \dot{e}_3 &= -p e_1 - a p e_2 + c p e_3 - p(1+d) e_4 + d_3, \\ \dot{e}_4 &= c p e_2 + (c p + d) e_4 + d_4. \end{aligned} \tag{6}$$

In other words, we have

$$\begin{pmatrix} e_1 \\ e_2 \\ e_4 \end{pmatrix} = \mathrm{e}^{At} \left[\begin{pmatrix} e_1(0) \\ e_2(0) \\ e_4(0) \end{pmatrix} + \int_0^t \mathrm{e}^{-As} \begin{pmatrix} d_1 \\ d_2 \\ d_4 \end{pmatrix} \mathrm{d}s \right],$$

where $A = \begin{bmatrix} 0 & p-1 & p \\ 1 & a & 1 \\ 0 & cp & cp+d \end{bmatrix}$ has the characteristic polynomial

$$\begin{aligned} f(\lambda) = \lambda^3 &- (cp+d+a)\lambda^2 \\ &+ (acp+ad-cp+1-p)\lambda + (dp-cp-d). \end{aligned}$$

By the Routh-Hurwitz theorem, the real parts of all its characteristic roots are negative if and only if

$$\begin{aligned} \Delta_1 &= -(cp+d+a) > 0, \\ \Delta_2 &= -(cp+d+a)(acp+ad-cp+1-p) \\ &\quad - (dp-cp-d) > 0, \\ \Delta_3 &= (dp-cp-d) > 0. \end{aligned} \tag{7}$$

Clearly, a constant p exists that satisfies the above conditions for any given parameters a, b, c, and d such that system (1) is hyperchaotic. Therefore, there exists positive constants α and β such that $|\mathrm{e}^{At}x| \le \alpha \mathrm{e}^{-\beta t}|x|$ for every $x \in \mathbb{R}^3$ and $t \ge 0$. Hence, we have

$$|e_i| \le \alpha \mathrm{e}^{-\beta t} \max_{i=1,2,4} |e_i(0)| + \frac{\alpha\delta}{\beta} \quad i = 1,2,4, \tag{8}$$

and thus

$$\overline{\lim_{t\to\infty}} |e_i(t)| \le \frac{\alpha}{\beta}\delta \quad i = 1,2,4. \tag{9}$$

Furthermore, since

$$\begin{aligned} |e_3| &= |-p(e_2+e_4)| \\ &\le 2|p| \left(\alpha \mathrm{e}^{-\beta t} \max_{i=1,2,4} |e_i(0)| + \frac{\alpha\delta}{\beta} \right) \end{aligned}$$

on the sliding mode surface $S = e_3 + p(e_2+e_4) = 0$, we have

$$\overline{\lim_{t\to\infty}} |e_3(t)| \le \frac{2|p|\alpha}{\beta}\delta. \tag{10}$$

Let $q = \frac{\alpha}{\beta}\max(1, 2|p|)$. The following theorem and corollary are immediate consequences of (8) and (9).

Theorem 2. If the controller is selected as

$$\begin{aligned} v = &-x_1e_3 - x_3e_1 - e_1e_3 - pe_1 - ape_2 \\ &+ cpe_3 - p(1+d)e_4 - k\,\mathrm{sgn}(S), \end{aligned}$$

where p is a constant satisfying the inequalities (7) and $k > (2|p|+1)\delta + 1$, then there exists a constant q such that the controlled Rössler hyperchaotic system with noise perturbation (2) is synchronous with the Rössler hyperchaotic system (1) with ultimate error bound $q\delta$, i. e. $\overline{\lim}_{t\to\infty}|y_i - x_i| \le q\delta$ $(i = 1,2,3,4)$.

Corollary 1. Under the conditions of Theorem 2, the controlled Rössler hyperchaotic system with noise perturbation (2) is synchronous with the Rössler hyperchaotic system (1).

4. Improving the Performance of the Sliding Mode Control

The sliding mode controller of Section 3 contains the discontinuous nonlinear function $\mathrm{sgn}(S)$ that could raise both theoretical and practical issues. Theoretically, the existence and uniqueness of solutions as well as their validity for Lyapunov analysis must all be examined in a framework that requires no local Lipschitz conditions on the right-hand-side functions of the state equation. Practical issues include chattering caused by imperfections in switching devices and delays, chattering results in low control accuracy, excessive heat losses (in electrical power circuits), and the wear and tear of moving mechanical parts. Chattering may also induce unwanted high-frequency dynamics that degrades the performance of the system and may even lead to instability.

One way of eliminating chattering is to replace the discontinuous sliding mode controller by a continuous approximation, which also avoids all the theoretical difficulties associated with discontinuous controllers. We therefore approximate the signum nonlinearity by a 'saturation' nonlinearity with great slope. In other words, the sliding mode controller v is taken to be

$$\begin{aligned} v = &-x_1e_3 - x_3e_1 - e_1e_3 - pe_1 - ape_2 \\ &+ cpe_3 - p(1+d)e_4 - k\,\mathrm{sat}\left(\frac{S}{\varepsilon}\right), \end{aligned} \tag{11}$$

where $\mathrm{sat}(\cdot)$ is the saturation function defined by

$$\mathrm{sat}(y) = \begin{cases} y & \text{if } |y| \le 1, \\ \mathrm{sgn}(y) & \text{if } |y| > 1, \end{cases}$$

and ε is a positive constant. The saturation nonlinearity $\mathrm{sat}\left(\frac{S}{\varepsilon}\right)$ then approaches the signum nonlinearity sign(S) in the limit as $\varepsilon \to 0$. Now, we examine the approaching control by using the Lyapunov function

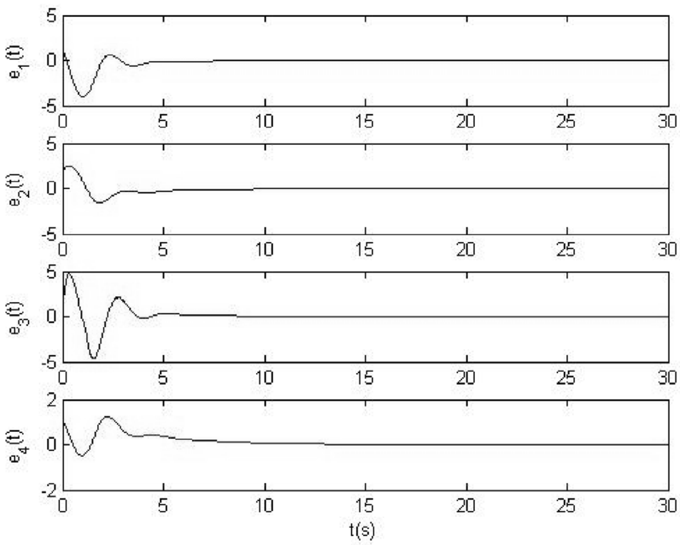

Fig. 1. Errors obtained by sliding mode control between the master and slave systems without disturbances.

$V = S^2$. Obviously, the derivative of V along the solution of system (3) satisfies the equality

$$\dot{V} = 2S\left[d_3 + p(d_2+d_4) - k\,\mathrm{sat}\left(\frac{S}{\varepsilon}\right)\right] \tag{12}$$

so that if $k > (2|p|+1)\delta + 1$ in the region $|S| > \varepsilon$, then we have

$$\begin{aligned}\dot{V} &\le 2|S|[|d_3| + |p|(|d_2| + |d_4|) - k]\\ &\le -2|S| = -2V^{\frac{1}{2}}.\end{aligned} \tag{13}$$

It follows that whenever $|S(0)| > \varepsilon$, $|S|$ will be strictly decreasing until it reaches the set $\{|S| \le \varepsilon\}$ in finite time and remain inside thereafter. In the bounded layer $\{|S| \le \varepsilon\}$, however, we have

$$\begin{aligned}\dot{V} &\le 2(2|p|+1)\delta|S| - \frac{2k}{\varepsilon}S^2\\ &= 2(2|p|+1)\delta V^{\frac{1}{2}} - \frac{2k}{\varepsilon}V,\end{aligned}$$

which implies that

$$V^{1/2} \le V^{1/2}(0)\mathrm{e}^{-\varepsilon t/k} + \frac{\varepsilon}{k}(2|p|+1)\delta \tag{14}$$

so that the error system in the boundary layer reads

$$\begin{aligned}\dot{e}_1 &= -e_2 - e_3 + d_1,\\ \dot{e}_2 &= e_1 + ae_2 + e_4 + d_2,\\ \dot{e}_3 &= -pe_1 - ape_2 + cpe_3 - p(1+d)e_4 + d_3 - \frac{k}{\varepsilon}S,\\ \dot{e}_4 &= -ce_3 + de_4 + d_4.\end{aligned}$$

The following theorem can be proved in the same way as Theorem 2 by using the inequality (14).

Theorem 3. If the controller is selected as (11), where p is a constant satisfying the inequalities (7) and $k > (2|p|+1)\delta + 1$, then there exists a constant q such that the controlled Rössler hyperchaotic system with noise perturbation (2) is synchronous with the Rössler hyperchaotic system (1) with ultimate error bound $q\delta$, i. e. $\overline{\lim}_{t\to\infty}|y_i - x_i| \le q\delta$ $(i = 1,2,3,4)$.

It is obvious from this theorem that the controlled Rössler hyperchaotic system without noise perturbation is synchronous with the Rössler hyperchaotic system (1). It is worth mentioning here that the controller in Theorem 3 can be implemented because it is continuous and there are no chattering phenomena.

5. Numerical Simulations

In order to verify the validity of the proposed method, we will present some numerical simulations in this section. The parameters of the Rössler hyperchaotic system are selected to be $a = 0.25$, $b = 3$, $c = 0.5$, and $d = 0.05$ in all the simulations. The initial states of the master and slave systems are $(-20,-1,0,15)^{\mathrm{T}}$ and $(-19,1,1,16)^{\mathrm{T}}$, respectively, and the constant in the sliding mode is selected to be $p = -5$. The constants in the sliding mode controller are selected to be $k = 12$ and $\varepsilon = 0.01$. The results of the numerical simulations are shown in Figures 1 – 5.

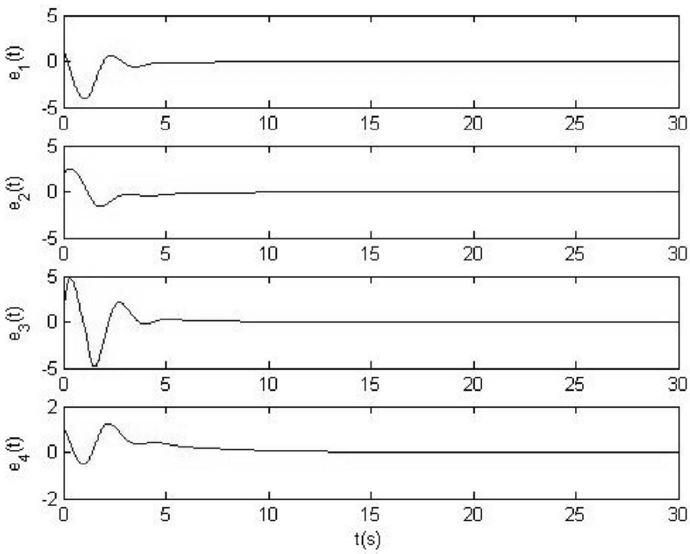

Fig. 2. Errors obtained by sliding mode control between the master and slave systems with disturbances $d_1 = 0$, $d_2 = 0$, $d_3 = 0.5\sin(2t)$, and $d_4 = 0$.

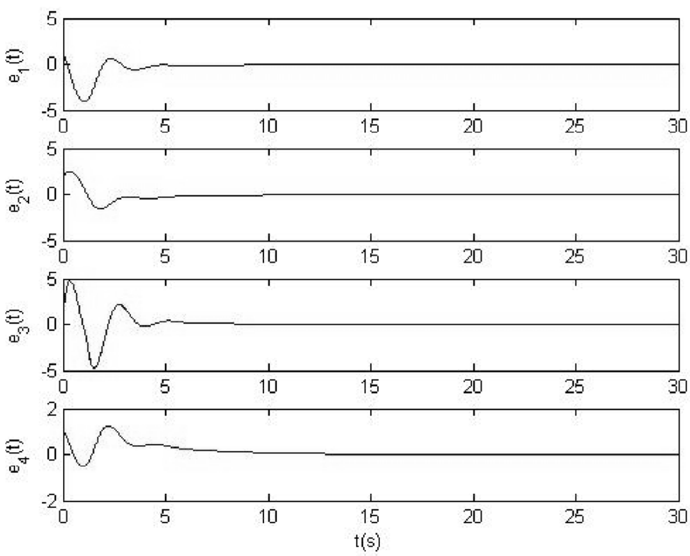

Fig. 3. Errors obtained by sliding mode control between the master and slave systems with disturbances $d_1 = 0$, $d_2 = 0.025\sin t$, $d_3 = 0.5\sin(2t)$, and $d_4 = 0$.

The numerical simulation results are clearly consistent with the theoretical analysis.

6. Conclusion

A simple linear sliding mode control scheme was presented in this paper to investigate the synchronization problem between two unidirectionally-coupled Rössler hyperchaotic systems with noise perturbations. The two hyperchaotic systems were synchronized by using a single and specifically designed controller. Some numerical simulation results were included to demonstrate the effectiveness and feasibility of the developed approach.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 61070087), the National Natural Science Foundation of Guangdong province (Grant No. 2008329), and a research fund from the Hong Kong Polytechnic University.

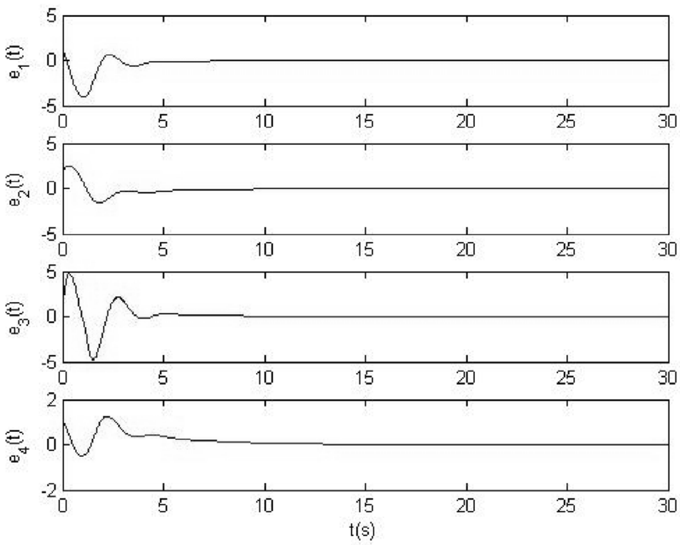

Fig. 4. Errors obtained by sliding mode control between the master and slave systems with disturbances $d_1 = 0.025\sin(t)$, $d_2 = 0$, $d_3 = 0.5\sin(2t)$, and $d_4 = 0.0125\sin(t)$.

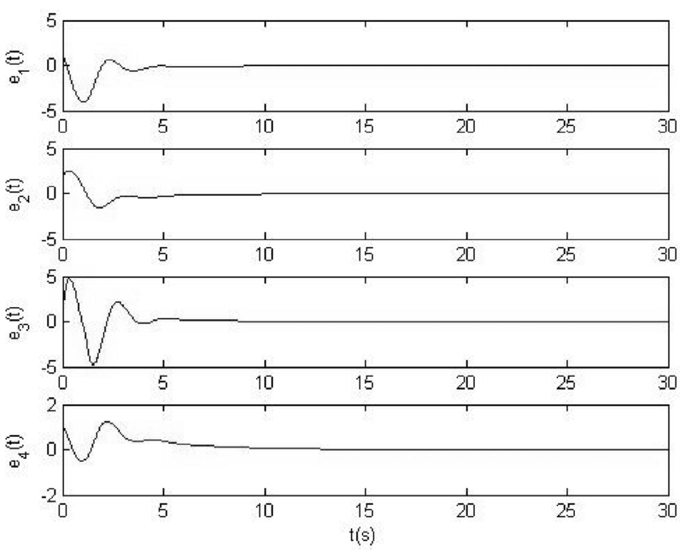

Fig. 5. Errors obtained by sliding mode control between the master and slave systems with disturbances $d_1 = 0.025\sin(t)$; $d_2 = 0.025\sin(t)$; $d_3 = 0.5\sin(t)$, and $d_4 = 0.0125\sin(2t)$.

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